Rigid Geometry and Applications II

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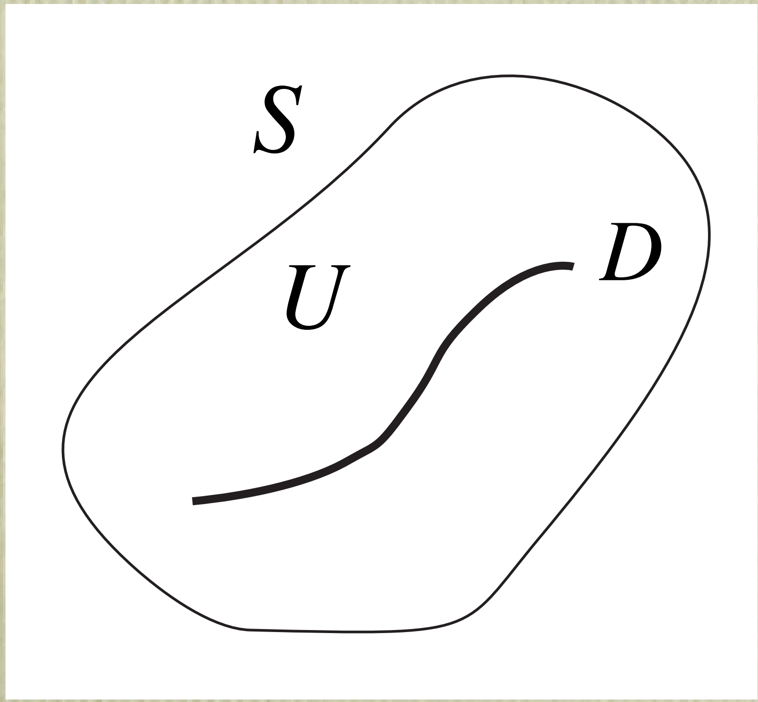
Birational Geometry from Zariski's viewpoint



- S: coherent (= quasi-compact and quasi-separated) (analog. compact Hausdorff)

- $U = S \setminus D \hookrightarrow S$ quasi-compact open immersion (U: dense in S).

$\mathcal{I}_D$ = defining ideal of D

Basic Question: Extension problem

- P: a property of morphisms (e.g. $P =$"flat")
- $f_U : X_U \longrightarrow U$: finitely presented $/U$ with property P.
- Assume $\exists\, f : X \to S$: an extension of f_U.

$\rightsquigarrow$ Can one find such an f with P ?

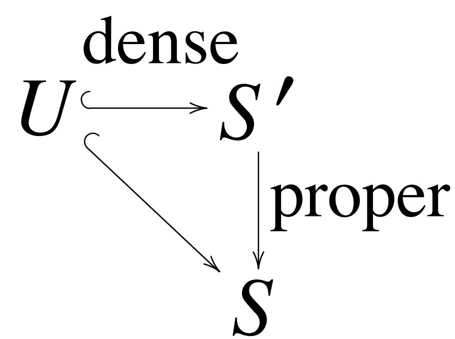
NO in general.

Need to allow birational changes of S preserving U.

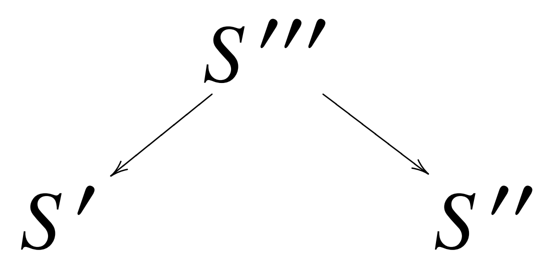
Modifications

$\mathbf{MD}_{(S,U)}$: Category of "U-admissible" modifications:

- Objects:



- Morphisms:



The category $\mathbf{MD}_{(X,U)}$ is cofiltered.

$\mathbf{BL}_{(S,U)}$: Full subcategory of "U-admissible" blow-ups:

- Objects: $S' \longrightarrow S$: a blow-up with the center $\subseteq S \setminus U$ (set-theoretically).

Example: $S = \operatorname{Spec} A$, $D = V(I)$

$$\rightsquigarrow S' = \operatorname{Proj} \bigoplus_{n \geq 0} J^n,\ I^k \subseteq J \text{ for } \exists\, k.$$

Strict Transform

$S' \longrightarrow S$: U-admissible modification.

$f\colon X \longrightarrow S$: a morphism.

$$\begin{array}{ccccc} X & \longleftarrow & X_{S'} = X \times_S S' & \hookleftarrow & X' \\ \downarrow & & \downarrow{\scriptstyle f_{S'}} & \swarrow{\scriptstyle f'} & \\ S & \longleftarrow & S' & & \end{array}$$

$X_{S'} \hookleftarrow X'$: dividing out $\mathcal{I}_D$-torsions.

Modified Extension Problem

$f_U\colon X_U \longrightarrow U$: finitely presented $/U$ with the property P.

Assume $\exists\, f\colon X \to S$: an extension of f_U.

$\rightsquigarrow$ Can one find a U-admissible modification (resp. blow-up) $S' \to S$ such that the strict transform $f'\colon X' \to S'$ has P ?

Flattening Theorem

Case: P=flat

Theorem (Gruson-Raynaud, 1970).

$f_U : X_U \longrightarrow U$: flat, finitely presented

$\Longrightarrow \exists\, S' \to S$: U-admissible blow-up such that

$f' : X' \to S'$: flat, finitely presented.

Corollary.

$\mathbf{BL}_{(S,U)}$ is cofinal in $\mathbf{MD}_{(S,U)}$.

Comments on Flattening Theorem

- Clear if $S = \operatorname{Spec} V$, V: DVR.
- If V: valuation ring, flatness is clear, while finite presentation is rather difficult.

Proof - Revival of Zariski's idea

Zariski-Riemann space (Zariski, 1939)

$$\langle U \rangle_{\mathrm{cpt}} = \varprojlim_{S' \in \mathbf{BL}_{(S,U)}} S'$$

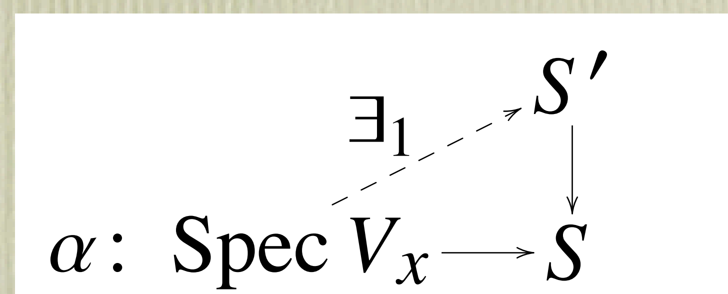
Projective limit taken in the category of local ringed spaces

(1) Generalization of abstract Riemann surface:
S: regular curve $\Longrightarrow \langle U \rangle_{\mathrm{cpt}} = S$ (Dedekind-Weber)
(2) Zariski's motivation: resolution of singularities

Points and local rings

$x \in \langle U \rangle_{\mathrm{cpt}} \supset U$

- $x \in U \implies \mathcal{O}_{\langle U \rangle_{\mathrm{cpt}},x} = \mathcal{O}_{U,x}$.
- $x \notin U \implies \exists\, V_x$: valuation ring (of height ≥ 1) with

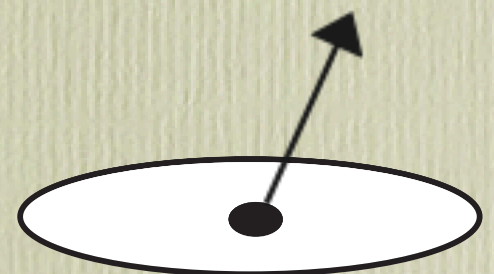


$\rightsquigarrow\ x = \{p_{S'}\}_{S' \in \mathbf{BL}_{(S,U)}}$: compatible system of points.

($p_{S'}$: the image of the closed points of Spec V_x.)

$\mathcal{O}_{\langle U\rangle_{\mathrm{cpt}}}$: $\mathcal{I}_D$-valuative ring

— A "composite" of local rings of U and valuation rings.



Notice: The valuation rings are not necessarily of height 1 (even when S: algebraic variety $/k$).

$\rightsquigarrow$ One has to consider valuation rings of *higher height*.

Example of valuation rings

S :algebraic surface

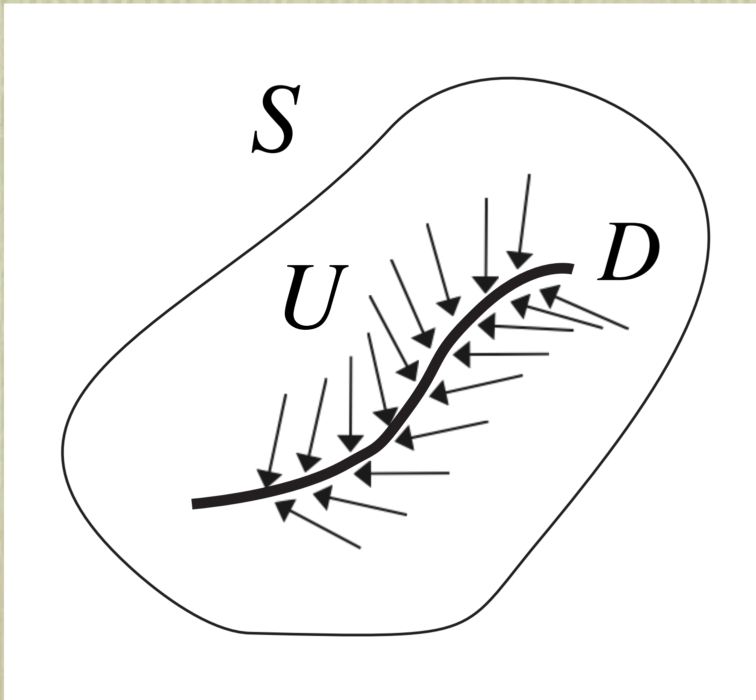
ht.	rat. rk.	
0	0	trivial valuation
1	1	divisorial
		non-divisorial
	2	non-divisorial
2	2	composite of two divisorial valuations

Intuitive Description

V: Valuation ring

$$\operatorname{Spec} V = \bullet \to \bullet \to \bullet \to \bullet \cdots \to \bullet$$

"Long Curve"



$$\langle U \rangle_{\mathrm{cpt}} = U \amalg \mathrm{T}^*_{D/S}$$

$\mathrm{T}^*_{D/S}$: The set of "long curves" passing through D: "*Tubular neighborhood*"

Fundamental Property

Theorem (Zariski, 1944).

$\langle U \rangle_{\text{cpt}}$ is quasi-compact.

- Flattening theorem: consequence of the quasi-compactness.

Proofs:

(1) $\varprojlim$ coherent topoi = coherent topos (cf. $\varprojlim$ cpt Hausdorff spaces = cpt Hausdorff space) + Deligne's theorem (existence of points)

(2) Stone's representation theorem for distributive lattices.

Proof of flattening theorem

Idea: Reduction to "long curves" using the quasi-compactness

(1) True for long curves, i.e. S=Spec V.

(2) True for S=Spec "local ring of a point of $\langle U \rangle_{\mathrm{cpt}}$.

(Follows from (1) and "true on U" by "patching".)

(1), (2), and P: locally finitely presented

$\Longrightarrow$ Claim is true "locally on $\langle U \rangle_{\mathrm{cpt}}$",

i.e., $\forall\ x \in \langle U \rangle_{\mathrm{cpt}}$,

$$\begin{array}{ccc} U_x & \hookrightarrow & S' \\ & & \downarrow \\ & & S \end{array}$$

***U*-Admissible blow-up**

Claim is true on U_x.

(3) Quasi-compactness $\Longrightarrow$ Done by birational patching

Other applications

- Embedding theorem for algebraic spaces (Nagata (1963): for Noetherian schemes).
- Resolution of singularities of quasi-excellent surfaces (Zariski).

Contents of the Book (Vol. I)

1. Preliminaries
2. Formal geometry
3. Basics on rigid geometry
4. Formal flattening theorem
5. Enlargement theorem

To be discussed (Vol. II) ?

1. Etale topology
2. Lefschetz trace formula
3. Relationship with logarithmic geometry & ramification theory

Why de we do this ?

1. Arithmetic geometry of Shimura varieties (local models, p-adic modular forms).

2. Cohomology theory of schemes (l-adic Lefschetz trace formula, rigid cohomology of Berthelot).

3. Mirror symmetry (expected construction of Mirror partner (Kontsevich, Fukaya,...)).

Recall: Our approach

Birational geometry of formal spaces = Rigid geometry

Raynaud's viewpoint of rigid geometry + Zariski's viewpoint of birational geometry

Geometry of models $\longrightarrow$ Rigid analytic geometry

Zariski-Riemann space

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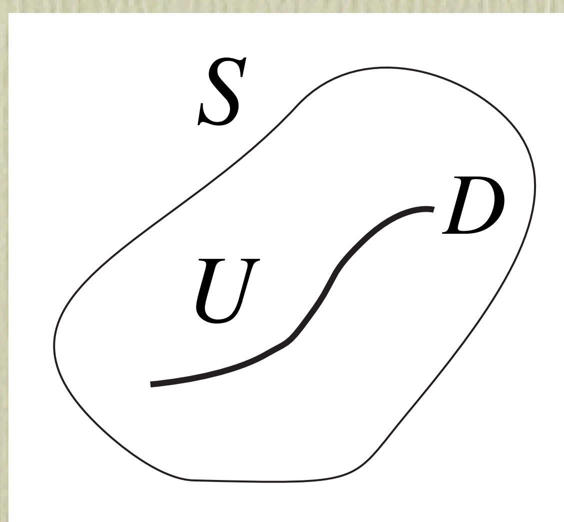
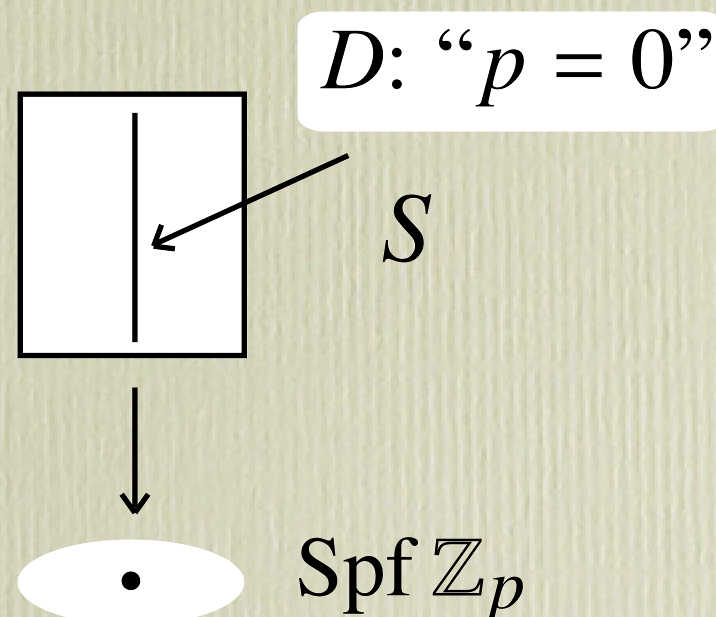
Geometry of Formal Schemes

Birational approach to Rigid Geometry

Birational Geometry	p-adic Rigid Geometry
	

Adequate Formal schemes

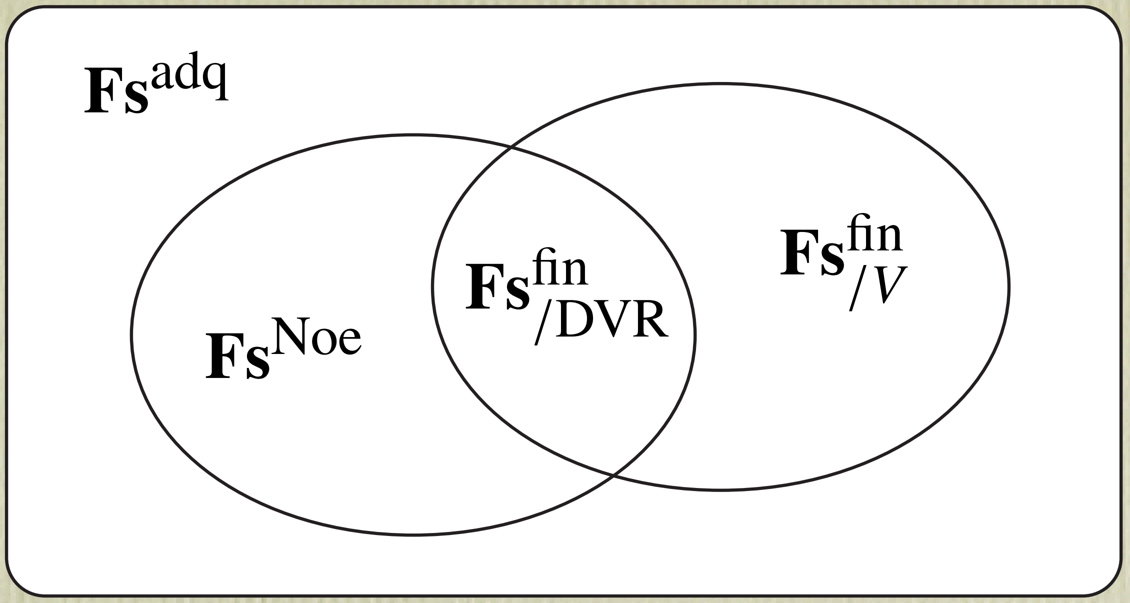
- Defined ring theoretically.
- Noetherian outside the ideal of definition.

Objects contain:

- Spf V, V: a-adically complete valuation ring.
- Noetherian formal schemes.

Functoriality:

- Closed under finite type extensions.
- Base change by finite type morphisms.



Notice:

- The height of V can be arbitrary.
- Includes formal models of Tate's affinoid algebras.

Nice Point: Can generalize Theorems in EGA III (Finitudes, GFGA Comparison, GFGA Existence) in derived categorical language.

Coherent Rigid Spaces

$\mathbf{CFs}^{\mathrm{adq}}$ = the category of coherent (= quasi-compact and quasi-separated) adequate formal schemes.

∃ Notion of "admissible blow-ups".

Definition. The category of coherent rigid spaces:

$$\mathbf{CRf} = \mathbf{CFs}^{\mathrm{adq}}/\{\text{admissible blow-ups}\}.$$

Quotient functor

$$\mathbf{CFs}^{\mathrm{adq}} \longrightarrow \mathbf{CRf} \qquad X \longmapsto X^{\mathrm{rig}}.$$

Admissible Topology

- $\mathcal{U} \hookrightarrow \mathcal{X}$: (coherent) open immersion $\Longleftrightarrow$ $\exists$ formal model $U \hookrightarrow X$: open immersion.
- $\{\mathcal{U}_\alpha \hookrightarrow \mathcal{X}\}$: (finite) covering $\Longleftrightarrow$ $\exists$ formal models $U_\alpha \hookrightarrow X$ such that $X = \bigcup U_\alpha$.

Proposition. Any representable presheaf on $\mathbf{CRf}_{\mathrm{ad}}$ is a sheaf.

(Proof: By birational patching of formal models.)

General Rigid Spaces

Definition.

General Rigid Space = a sheaf $\mathcal{F}$ of sets on $\mathbf{CRf}_{\mathrm{ad}}$ such that

(1) $\exists$ surjective map of sheaves

$$\coprod_{i \in I} \mathcal{Y}_i \longrightarrow \mathcal{F},$$

where $\mathcal{Y}_i$: coherent rigid spaces.

(2) For $i, j \in I$,

$$\mathcal{Y}_i \times \mathcal{Y}_j \longrightarrow \mathcal{Y}_i :$$

an increasing sequence of open immersions.

Rf = the category of general rigid spaces.

Relationship with algebraic spaces

Theorem. $\exists$ GAGA functor

$$\left\{\begin{matrix}\text{Algebraic spaces}\\ \text{of finite type }/\mathbb{Q}_p\end{matrix}\right\} \longrightarrow \mathbf{Rf}.$$

Ingredients for the proof:

- Embedding theorem for algebraic spaces.
- Equivalence Theorem:

Equivalence Theorem. S: coherent adequate formal scheme

$$\left\{\begin{array}{l}X/S\colon \text{ formal algebraic space of} \\ \text{finite type}\end{array}\right\} \Big/ \{\text{adm. blow-ups}\}$$

$$\xrightarrow{\sim} \left\{\begin{array}{l}X/S\colon \text{ formal scheme} \\ \text{of finite type}\end{array}\right\} \Big/ \{\text{adm. blow-ups}\}$$

Follows from:

Theorem. S: as above, X/S: formal algebraic space of finite type.
$\Longrightarrow \exists\, \pi\colon X' \to X$: admissible blow-up such that X': formal scheme.

Visualization

- $\mathcal{X} = X^{\mathrm{rig}}$: coherent rigid space

$$\langle \mathcal{X} \rangle = \varprojlim_{X' \to X} X'$$

(projective limit of all admissible blow-ups).

- Specialization map $\mathrm{sp}_{X'} \colon \langle \mathcal{X} \rangle \longrightarrow X'$.

- $\mathcal{O}^{\mathrm{int}}_{\mathcal{X}}$ (Integral Structure Sheaf): limit of $\mathcal{O}_{X'}$.

- Rigid Structure Sheaf:

$$\mathcal{O}_{\mathcal{X}} = \varinjlim_{n \geq 0} \mathcal{H}om_{\mathcal{O}^{\mathrm{int}}_{\mathcal{X}}}(\mathcal{I}^n, \mathcal{O}^{\mathrm{int}}_{\mathcal{X}}),$$

where $\mathcal{I}$: ideal of definition.

The ring used yesterday

In p-adic situation: $\mathcal{O}_{\mathcal{X}} = \mathcal{O}^{\mathrm{int}}_{\mathcal{X}}[\frac{1}{p}]$.

$\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ **is the canonical integral model of** $\mathcal{O}_{\mathcal{X}}$

$$\rightsquigarrow \text{Zariski-Riemann triple } \mathbf{ZR}(\mathcal{X}) = (\langle \mathcal{X} \rangle, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}, \mathcal{O}_{\mathcal{X}}).$$

Defined similarly for general rigid spaces

Remark. Notice the analogy:

$$(\mathcal{E}, |\ \ |) : \text{ Hermitian bundle}$$

$$\longleftrightarrow \quad \mathcal{E} \text{ with integral model } \mathcal{E}^{\mathrm{int}}.$$

Proposition.

Topos associated to $\langle \mathcal{X} \rangle \ \cong\ $ admissible topos $\mathcal{X}_{\mathrm{ad}}^{\sim}$.

Relation with other theories

(1) Tate's rigid spaces are naturally objects of **Rf** (via Raynaud's theorem & patching).

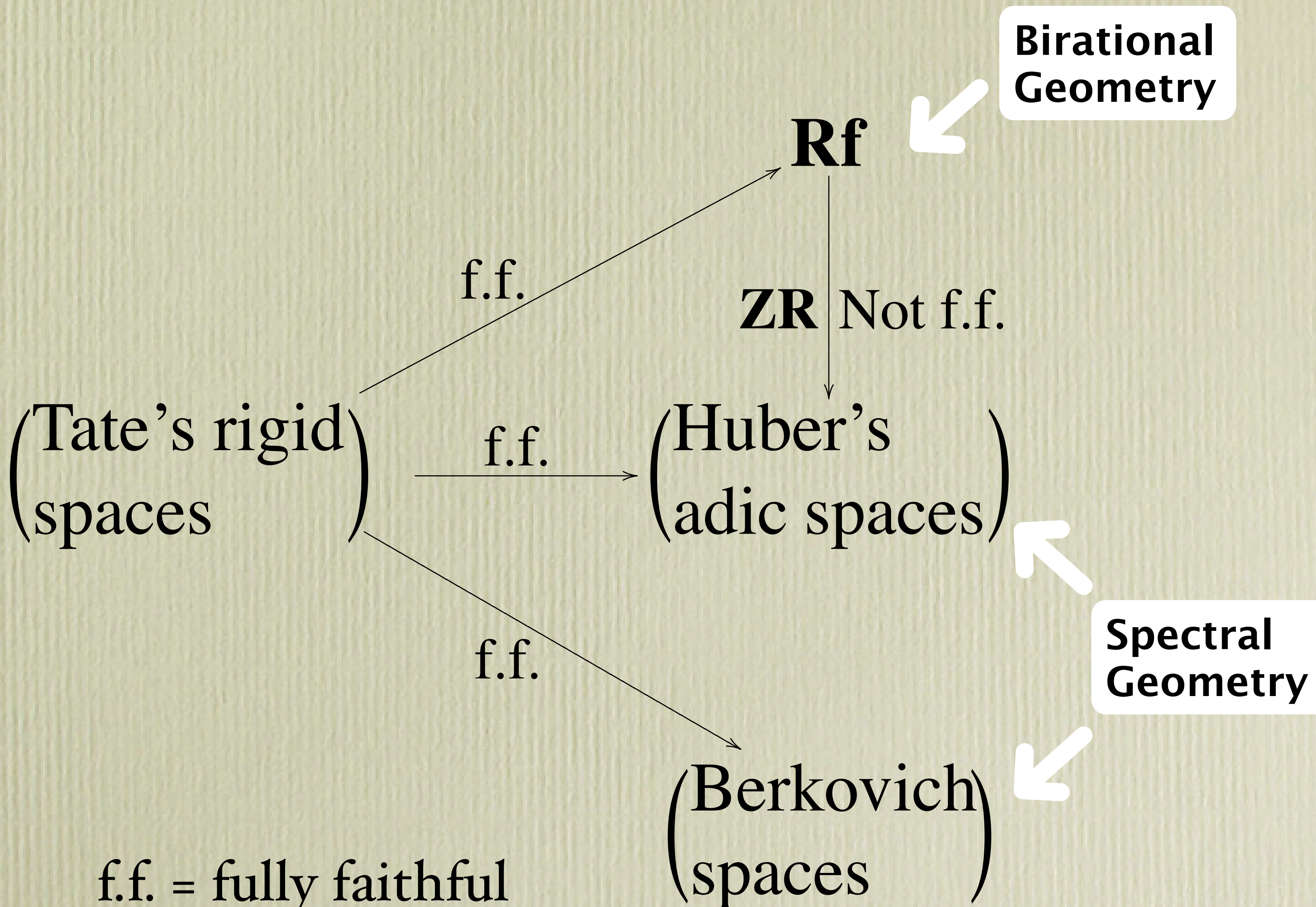
$$\begin{pmatrix}\text{Tate's rigid}\\ \text{spaces}\end{pmatrix} \longrightarrow \mathbf{Rf}$$

$$\text{affinoids} \longmapsto \text{affinoids.}$$

Affinoid in **Rf** = Coherent rigid space of the form $(\mathrm{Spf}\,A)^{\mathrm{rig}}$.

(2) Zariski-Riemann triples are regarded as Huber's adic spaces.

Diagram


Birational Geometry
Rf
f.f.
ZR
Not f.f.
Tate's rigid spaces
f.f.
Huber's adic spaces
Spectral Geometry
f.f.
Berkovich spaces
f.f. = fully faithful

Formal Flattening Theorem

Theorem (Bosch-Raynaud, Fujiwara).

- $f\colon X \to S$: finite type between coherent adequate formal schemes.

$\Longrightarrow$ the following conditions are equivalent:

(1) $f^{\mathrm{rig}}\colon X^{\mathrm{rig}} \to S^{\mathrm{rig}}$: flat ($\overset{\text{def}}{\Longleftrightarrow}$ $\langle f^{\mathrm{rig}}\rangle$: flat as a morphism of local ringed spaces).

(2) $\exists\, S' \to S$: admissible blow-up such that the strict transform $f'\colon X' \to S'$ is flat.

Proof: Similar to the algebraic case.

Corollaries

Corollary. Admissible blow-ups are cofinal in the category of formal modifications.

Corollary (Gerritzen-Grauert). In Tate's rigid analytic geometry,

$$\text{sep. étale inj.} \quad \Longleftrightarrow \quad \text{open imm.}$$

Properness in Rigid Geometry

Three different definitions

1. Universally closed, separated, of finite type.
2. Raynaud properness; i.e., having proper formal model.
3. Kiehl properness; i.e., existence of affinoid enlargements of coverings for each relatively compact affinoid open subset.

Difficult: 2. $\Longrightarrow$ 3.

For $f\colon X \to \operatorname{Spf} V$: of finite type,

Showing 2. $\Longrightarrow$ 3.

Theorem.
$U \subset X$: affine open such that $\overline{U}$: proper
$\Longrightarrow \exists\, \pi\colon X' \to X$: admissible blow-up and $\exists\, W \subset X'$: open such that

- $\overline{\pi^{-1}(U)} \subseteq W$,
- $W \to \operatorname{Spf} A$: contraction to an affine formal scheme.

Remark. Proved by Lütkebohmert, when $V = \mathrm{DVR}$ (1990).

Formal Geometry and Rigid Geometry

General Principle

Theorems in Formal Geometry	⟷	Theorems in Rigid Geometry

e.g.

GFGA theorems ⟷ GAGA theorems

Finiteness Theorem

Formal Geometry	Rigid Geometry
Y: quasi-compact adequate and amenable $f\colon X \to Y$: proper of finite presentation $\Longrightarrow$ $\mathrm{R}f_*$ maps $\mathbf{D}^*_{\mathrm{coh}}(X)$ to $\mathbf{D}^*_{\mathrm{coh}}(Y)$ for $* = \emptyset, +, -, \mathrm{b}$.	$\mathcal{Y}$: quasi-compact $\varphi\colon \mathcal{X} \to \mathcal{Y}$: proper $\Longrightarrow$ $\mathrm{R}\varphi_*$ maps $\mathbf{D}^*_{\mathrm{coh}}(\mathcal{X})$ to $\mathbf{D}^*_{\mathrm{coh}}(\mathcal{Y})$ for $* = \emptyset, +, -, \mathrm{b}$.

Comparison Theorem

Formal Geometry	Rigid Geometry
(Y, Z): universal adhesive pair, amenable $f\colon X \to Y$: proper of finite presentation $\Longrightarrow$ $\begin{array}{ccc} \mathbf{D}^*_{\mathrm{coh}}(X) & \xrightarrow{\text{for}} & \mathbf{D}^*(\widehat{X}) \\ \downarrow \mathrm{R}f_* & & \downarrow \mathrm{R}\widehat{f}_* \\ \mathbf{D}^*_{\mathrm{coh}}(Y) & \xrightarrow[\text{for}]{} & \mathbf{D}^*(\widehat{Y}) \end{array}$ (with $\approx$ on the double arrow from $\mathbf{D}^*_{\mathrm{coh}}(Y)$ to $\mathbf{D}^*(\widehat{Y})$)	(B, I): complete t.u.a., $U = \operatorname{Spec} B \setminus V(I)$. $f\colon X \to Y$: proper U-morphisms. $\Longrightarrow$ $\begin{array}{ccc} \mathbf{D}^*_{\mathrm{coh}}(X) & \xrightarrow{\text{rig}} & \mathbf{D}^*_{\mathrm{coh}}(X^{\mathrm{an}}) \\ \downarrow \mathrm{R}f_* & & \downarrow \mathrm{R}f^{\mathrm{an}}_* \\ \mathbf{D}^*_{\mathrm{coh}}(Y) & \xrightarrow[\text{rig}]{} & \mathbf{D}^*(Y^{\mathrm{an}}) \end{array}$ (with $\approx$ on the double arrow from $\mathbf{D}^*_{\mathrm{coh}}(Y)$ to $\mathbf{D}^*(Y^{\mathrm{an}})$)

Existence Theorem

Formal Geometry	Rigid Geometry
(B, I): complete t.u.a. pair, univ. top. coh. $f: X \to Y = \operatorname{Spec} B$: proper of finite presentation $\Longrightarrow$ $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{X})$	(B, I): complete t.u.a., $U = \operatorname{Spec} B \setminus V(I)$. $f: X \to Y$: proper U-morphisms. $\Longrightarrow$ $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{an}})$