Rigid geometry and Applications

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Applications

- Arithmetic geometry of Shimura varieties ($\boldsymbol{p}$-adic period map and local models, $\boldsymbol{p}$-adic automorphic representations)
- Cohomology theory of algebraic varieties ($\boldsymbol{\ell}$-adic Lefschetz trace formulas, $\boldsymbol{p}$-adic cohomology theories)

Possible applications

- Mirror symmetry
  (Construction of Mirror partner)
- $\boldsymbol{p}$-adic Hodge theory (via almost étale)
- Derived category equivalence

Rigid spaces vs schemes

Merit of using non-scheme theoretical geometric objects:

- Topological feature: Admissible topology is finer than Zariski topology
- Allow "infinite repetition construction" : non-coherent objects play an essential role (e.g. $\boldsymbol{p}$-adic uniformization)

Constant deformation technique

How to apply the rich structure to schemes:

Variety		Formal scheme		Rigid space
$\boldsymbol{X}$	$\rightsquigarrow$	$\widehat{\boldsymbol{X}}_{\boldsymbol{k}[[\boldsymbol{t}]]}$	$\rightsquigarrow$	$(\widehat{\boldsymbol{X}}_{\boldsymbol{k}[[\boldsymbol{t}]]})^{\mathbf{rig}}$
$\downarrow$		$\downarrow$		$\downarrow$
$\mathbf{Spec}\,\boldsymbol{k}$		$\mathbf{Spf}\,\boldsymbol{k}[[\boldsymbol{t}]]$		$(\mathbf{Spf}\,\boldsymbol{k}[[\boldsymbol{t}]])^{\mathbf{rig}}$

Applications

Finiteness of coherent cohomology for proper stacks

Theorem (Faltings). $\boldsymbol{S}$: *noetherian algebraic space*
$\boldsymbol{f : X \to S}$: *proper fppf-algebraic stack* $\boldsymbol{/S}$, $\boldsymbol{F}$: *coherent sheaf on* $\boldsymbol{X} \Rightarrow \boldsymbol{R^q f_* F}$: *coherent for* $\boldsymbol{q} \in \mathbb{Z}$.

“Rigid analytic” proof

For simplicity, assume X: algebraic variety $/k$.
$K = k((t))$. $\mathcal{X} = X_K^{an}$, $\mathcal{F} = F_K^{\mathrm{rig}}$.

<u>Grauert-Kiehl method</u>:

$\exists$ Finite affinoid coverings $\{\mathcal{U}_i\}_{i \in I}$, $\{\mathcal{V}_i\}_{i \in I}$ such that

- $\mathcal{U}_i$ is relatively compact in $\mathcal{V}_i$, i.e., $\mathcal{V}_i$ is an affinoid enlargement of $\mathcal{U}_i$.

Existence of the coverings $\Leftarrow$ Kiehl properness.

Affinoid covers: Leray covering (theorem A for affinoids)

$\rightsquigarrow$ Čech calculation:

$$\boldsymbol{H^q(\mathcal{X}, \mathcal{F}) \simeq H^q(C(\{\mathcal{V}_i\}_{i\in I}, \mathcal{F}))}$$

$$\boldsymbol{\simeq H^q(C(\{\mathcal{U}_i\}_{i\in I}, \mathcal{F}))}$$

(isom. of $\boldsymbol{K}$-Banach spaces)

Especially, every cohomology class is overconvergent.

Key Claim. *Induced isomorphism*

$$\boldsymbol{H}^q(\boldsymbol{C}(\{\boldsymbol{\mathcal{V}}_i\}_{i\in \boldsymbol{I}}, \boldsymbol{\mathcal{F}})) \simeq \boldsymbol{H}^q(\boldsymbol{C}(\{\boldsymbol{\mathcal{U}}_i\}_{i\in \boldsymbol{I}}, \boldsymbol{\mathcal{F}}))$$

is $\boldsymbol{K}$*-linearly compact.*

Unit ball is relatively compact $\Rightarrow$ finiteness.

Relative compactness of $\mathcal{U}_i \hookrightarrow \mathcal{V}_i$

$$\Longrightarrow \text{Key claim.}$$

Example. Enlargement

$\mathbb{D}^1(0, |a|) \hookrightarrow \mathbb{D}^1(0, 1)$, $\mathcal{F} = \mathcal{O}$.

The corresponding homomorphism

$$K\langle\langle X\rangle\rangle \to K\langle\langle \frac{X}{a}\rangle\rangle$$

is K-linearly compact. For example, $\{X^n\}_{n\in\mathbb{N}}$ is mapped to $\{a^n(\frac{X}{a})^n\}_{n\in\mathbb{N}}$, which converges to 0.

To conclude, need to deduce the finiteness $/\boldsymbol{k}$ from the rigid statement$/\ \boldsymbol{k}((\boldsymbol{t}))$.

Frobenius

- S: $\mathbb{F}_q$-scheme,
- $\mathbf{Fr}_q : S \longrightarrow S$: Frobenius $/\mathbb{F}_q = q$-th power map,
- $\mathcal{C}_S$: some category of geometric objects$/S$

The importance of $\mathbf{Fr}_q$: (usually) induces a self-map

$$\mathbf{Fr}_q^* : \mathcal{C}_S \longrightarrow \mathcal{C}_S.$$

i.e, induces *self-similarity*.

$\mathbf{Fr}_{\boldsymbol{q}}$-structure $\mathbf{Fr}_{\boldsymbol{q}}^{*}\boldsymbol{A} \simeq \boldsymbol{A}$

$\longleftrightarrow$ an $\mathbf{Fr}_{\boldsymbol{q}}$-fixed point.

Q: What happens near $\mathbf{Fr}_{\boldsymbol{q}}$-fixed point?

Constant deformation and Frobenius

Observation.

$\mathbb{A}^{1}_{\mathbb{C}} \longrightarrow \mathbb{A}^{1}_{\mathbb{C}}$, $X \longmapsto X^{q}$ is contracting near $\mathbf{0}$ for analytic topology

Rigid geometry situation.

$\mathbb{D}^{1} \longrightarrow \mathbb{D}^{1}$, $X \longmapsto X^{q}$: contracting near $\mathbf{0}$

$(\mathbf{Fr}_{q}(\mathbb{D}^{1}(r)) \subset \mathbb{D}^{1}(r^{q}))$

To treat $\mathbb{F}_q$: Use constant deformation

$$\mathbb{F}_q \longrightarrow \mathbb{F}_q[[t]]$$

$$X \longrightarrow X_{\mathbb{F}_q((t))} \longrightarrow \mathcal{X} = (X_{\mathbb{F}_q((t))})^{\mathbf{rig}}$$

$Y \subset X$: $\mathbb{F}_q$-invariant subspace

$\mathcal{Y} = (Y_{\mathbb{F}_q((t))})^{\mathbf{rig}}$

Regard $\mathbf{Fr}_q$ as a dynamical system

$$\mathcal{X} \xrightarrow{\mathbf{Fr}_q} \mathcal{X} \xrightarrow{\mathbf{Fr}_q} \mathcal{X} \xrightarrow{\mathbf{Fr}_q} \cdots$$

Claim. *"Frobenius is contracting near Y", i.e., $\mathbf{Fr}_q$ is contracting near $\langle \mathcal{Y} \rangle$ in $\langle \mathcal{X} \rangle$.*

Trace Formula in characteristic $\boldsymbol{p}$

This principle appeared in the study of Lefschetz trace formula in characteristic $\boldsymbol{p}$ (Solution of Deligne's conjecture, [F], Invent. Math.)

Situation.

X/k: algebraic space

$a : Y \to X \times_k X$: correspondence (a_1 proper, a_2 quasi-finite)

K: $\bar{\mathbb{Q}}_\ell$-complex ($\frac{1}{\ell} \in k$) with cohomological correspondence compatible with a.

Lefschetz number $\in \bar{\mathbb{Q}}_\ell$ is defined by

$$\mathrm{Lef}(a, R\Gamma_c(X, K)) = \mathrm{Trace}(a^*, R\Gamma_c(X, K)).$$

Theorem (Deligne's conjecture). *For* $\boldsymbol{k} = \bar{\mathbb{F}}_{\boldsymbol{q}}$, *if the data admit* $\mathbf{Fr}_{\boldsymbol{q}}$*-structure,* $\exists \boldsymbol{N} \in \mathbb{N}$ *s.t.*

- $\dim \mathbf{Fix}(\mathbf{Fr}_q^n \circ \boldsymbol{a}) = \mathbf{0}$ *for* $\boldsymbol{q}^{\boldsymbol{n}} > \boldsymbol{N}$.
- *For* $\boldsymbol{q}^{\boldsymbol{n}} > \boldsymbol{N}$,

$$\mathbf{Lef}(\mathbf{Fr}_q^n \circ \boldsymbol{a}, \boldsymbol{R}\Gamma_{\boldsymbol{c}}(\boldsymbol{X}, \boldsymbol{K})) = \sum_{D \in \mathbf{Fix}(\mathbf{Fr}_q^n \circ a)} \mathbf{naive.loc}_{D}(\mathbf{Fr}_q^n \circ \boldsymbol{a}, \boldsymbol{K}).$$

Here $\mathbf{naive.loc}_{D}(\boldsymbol{a}, \boldsymbol{K})$ vanishes if $\boldsymbol{K}|_{D} = \mathbf{0}$.

Structure of proof: Establish trace formula for rigid analytically contracting correspondence

Note: Not completely scheme-theoretical.

Other approaches:

$\boldsymbol{X}$: smooth algebraic var., $\boldsymbol{K}$: smooth sheaf:
Shpiz, Pink (arround 1990): Under $\exists$ good compactification and tameness of $\boldsymbol{K}$
Recent scheme-theoretic proof by T. Saito

Applications of Deligne's conjecture:

- Non-abelian class field theory (Shtuka moduli (Lafforgue, need more general trace formula), Shimura varieties (Harris-Taylor, ...))
- Representation theory of Chevalley groups (Digne-Rouquier, ...)
- Model theory (Hrushovski-Macintyre)

Questions:

- Deligne's conjecture for overconvergent $\boldsymbol{F}$-crystals.
- Rigid analytic technique in Mori theory

Number theoretical examples

Analysis of arithmetic moduli

$\boldsymbol{\mathcal{M}}$ = the moduli space of elliptic curves $/\mathbb{Z}$

($\boldsymbol{\mathcal{E}}^{\boldsymbol{univ}}$: Universal curve)

We view $\boldsymbol{\mathcal{M}}$ as an example of Shimura varieties.

Analysis near cusps

Universal Tate curve construction

$\boldsymbol{N}$ = the moduli space of 1-motives

$\boldsymbol{L} = [\mathbb{Z} \longrightarrow \mathbb{G}_{\boldsymbol{m}}] \sim \mathbf{Spec}\,\mathbb{Z}[\boldsymbol{q}, \boldsymbol{q}^{-1}],\ \mathbf{1} \mapsto \boldsymbol{q}$

(mixed Shimura variety)

$\bar{\boldsymbol{N}} \sim \mathbf{Spec}\,\mathbb{Z}[\boldsymbol{q}]$ (partial compactification of $\boldsymbol{N}$)

$\boldsymbol{S} = \bar{\boldsymbol{N}}^{\boldsymbol{cs}} \sim \mathbf{Spec}\,\mathbb{Z}[[\boldsymbol{q}]],\ \boldsymbol{D} = \{\boldsymbol{q} = \mathbf{0}\},$

$\boldsymbol{U} = \boldsymbol{S} \setminus \boldsymbol{D}.$

$A = \mathbb{G}_m / q^{\mathbb{Z}}$: Tate curve on S

A_U: elliptic curve, $A_D = \mathbb{G}_m$ (A: semi-abelian).

$\Rightarrow$ We have classifying map

$$\alpha : U \to \mathcal{M}, \quad \alpha^* \mathcal{E}^{univ} = A_U.$$

Arithmetic compactification

Theorem. $\exists$ *proper smooth algebraic stack* $\overline{\boldsymbol{\mathcal{M}}}/\mathbb{Z}$ *containing* $\boldsymbol{\mathcal{M}}$ *as an open substack s.t. the completion of* $\overline{\boldsymbol{\mathcal{M}}}$ *along infinity is isomorphic to* $\boldsymbol{S} = \bar{\boldsymbol{N}}^{\boldsymbol{cs}} \sim \mathbf{Spec}\,\mathbb{Z}[[\boldsymbol{q}]]$.

- The construction is generalized to more general Shimura varieties (Hilbert-Blumenthal (Rapoport), Siegel modular (Faltings-Chai), PEL-type (F.))
- Logarithmic approach representing functors: K. Kato, Nakayama, Kajiwara.

Applications:

- Integrality of holomorphic modular forms, e.g.,

$$j(q) = q^{-1} + \sum_{n \geq 0} a_n q^n$$

  $a_n \in \mathbb{Z}$ from the definition of j.

- Congruences between holomorphic modular forms (Construction of p-adic L-function (Deligne-Ribet), Main conjecture for elliptic curves (Skinner-Urban))

Construction

Patch $\mathcal{M}$ and S along U via α.

First step: Show that α is formally étale.

- Take a zero-dimensional closed subscheme $Z \subset U$ supported at a closed point x.
- Apply infinitesimal criterion for formal étaleness at x. Reduced to:

  Any deformation of $A_x = \mathbb{G}_m/q_x^{\mathbb{Z}}$ as an elliptic curve comes from A_U.

- Uniformization theory on $\boldsymbol{Z} \Rightarrow$ deformation of $\boldsymbol{A_x} =$ deformation of 1-motive $[\mathbb{Z} \longrightarrow \mathbb{G}_{\boldsymbol{m}}]$ $(\mathbf{1} \mapsto \boldsymbol{q_x})$, hence comes from $\boldsymbol{U}$.

Second step: Algebraize $(\boldsymbol{A}, \boldsymbol{S})$ by approximation theorem: We have a semi-abelian family $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{S}})$ which are of finite type over $\mathbb{Z}$. $\tilde{\boldsymbol{D}} =$ the locus where $\tilde{\boldsymbol{A}}$ is not an elliptic curve. Then

- $\hat{\hat{\tilde{\boldsymbol{S}}}}|_{\tilde{\boldsymbol{D}}} \simeq \hat{\boldsymbol{S}}|_{\boldsymbol{D}}$.
- The classifying map $\tilde{\boldsymbol{S}} \setminus \tilde{\boldsymbol{D}} \longrightarrow \boldsymbol{\mathcal{M}}$ is étale.

Final step: Patch $\boldsymbol{\mathcal{M}}$ and $\tilde{\boldsymbol{S}}$ along $\tilde{\boldsymbol{S}} \setminus \tilde{\boldsymbol{D}}$ by extending the equivalence relation.

Properness assured by Grothendieck's semi-stable reduction theorem for abelian varieties.

p-adic overconvergent forms

p: fixed, $X = \overline{\mathcal{M}}/\mathbb{Z}_p$.

$U = X_{\mathbb{F}_p}^{ord} \subset \mathcal{M}/\mathbb{Z}_p \subset X_{\mathbb{F}_p}$: the ordinary locus. We view U as an open subformal space of $\hat{X}$. $\mathcal{U} = U^{\mathrm{rig}}$, $\mathcal{X} = (\hat{X})^{\mathrm{rig}}$: The associated p-adic rigid spaces.

Definition.

- *The space* $\mathcal{S}_k$ *of* p*-adic modular forms of weight* $k = \Gamma(\mathcal{U}, \omega^k|_{\mathcal{U}})$
- *The space* $\mathcal{S}_k^{ov}$ *of overconvergent* p*-adic modular forms of weight* $k = \Gamma(\overline{\mathcal{U}}, \omega^k|_{\overline{\mathcal{U}}})$

$\mathcal{S}_k^{ov} \subset \mathcal{S}_k$.

Analysis near cusps

$x = \infty_{\mathbb{F}_p}$: cusp in $X_{\mathbb{F}_p}$.

The tube C_x of $x = \mathbf{sp}_{\hat{X}}^{-1}(x)^{int}$: admissible open subset of $\mathcal{U}$.

C_x depends only on $\hat{X}_x$. By the local description at cusp, $C_x = \{|q| < 1\}$.

$C_x \setminus \infty_{\mathbb{Q}_p} \subset \mathcal{U}$.

$\Rightarrow$ We get rigid analytic q-expansion of overconvergent forms.

By geometric reasons, $\boldsymbol{U_p}$: $\boldsymbol{\mathcal{S}_k^{ov}} \longrightarrow \boldsymbol{\mathcal{S}_k^{ov}}$ is a compact operator $\Rightarrow$ Coleman-Mazur "eigencurve"

Cf. General construction of eigenvariety due to Emerton, Urban