Moduli of regular holonomic $\mathcal{D}_X$-modules with natural parabolic semi-stability

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Riemann-Hilbert correspondence is an **equivalence of categories** –

Category of Regular Holonomic $\mathcal{D}$-modules.

Category of Perverse sheaves.

De Rham **functor**.

The basic problem is to **geometrize** the Riemann-Hilbert correspondence.

Moduli space for regular holonomic $\mathcal{D}$-modules.

Moduli space for perverse sheaves.

Riemann-Hilbert **morphism**.

Higgs objects as third vertex of the correspondence.

$\mathcal{D}$-modules

X nonsingular complex projective variety.

$\mathcal{D}_X$ the sheaf of linear partial differential operators on X (in the algebraic category).

$\mathcal{O}_X \hookrightarrow \mathcal{D}_X$ ring homomorphism (scalar operators). $T_X \subset \mathcal{D}_X$ (left $\mathcal{O}_X$-submodule).

$\mathcal{D}_X$ generated by $\mathcal{O}_X$ and T_X.
Relations (for $f \in \mathcal{O}_X$ and $\xi, \eta \in T_X$) :
$\xi f - f\xi = \xi(f)$, and $\xi\eta - \eta\xi = [\xi, \eta]$.

$\mathcal{D}_X$ has increasing filtration $F^i\mathcal{D}_X$ by order. Graded algebra $Sym^{\bullet}_{\mathcal{O}_X} T_X$.

Left $\mathcal{D}_X$-modules, $\mathcal{O}_X$-quasi-coherent.

Holonomicity and regularity conditions on M, and characteristic variety $car(M) \subset T^*X$: alternative concrete formulations given later.

Geometric set up

We fix a normal crossing divisor $Y \subset X$.

Let $S_d = X$ and $S_{d-1} = Y$ where $d = dim(X)$.

S_{i-1} = singular locus of S_i for $i \leq d-1$.

N_i^* = co-normal bundle of $S_i - S_{i-1}$ in $X - S_{i-1}$.

$T^*X \supset C_{X,Y} = N_d^* \cup N_{d-1}^* \cup \ldots$

We consider all regular holonomic M for which characteristic variety $car(M) \subset C_{X,Y}$.

Corresponding perverse sheaves: all those which are cohomologically constructible w.r.t the stratification $X = \coprod_i (S_i - S_{i-1})$.

Integrable Connections

Simplest case of regular holonomic modules: Pairs $M = (E, \nabla)$ where E is a vector bundle on X and $\nabla : E \to \Omega^1 \otimes E$ is a connection which is integrable.

A local section ξ of T_X acts by ∇_ξ on E, making it a $\mathcal{D}_X$ module because:

∇_ξ is $\mathcal{O}_X$-linear in ξ,
$\nabla_\xi(fv) = \xi(f)v + f\nabla_\xi v$ (Leibniz rule), and
$[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$ (integrability).

A left $\mathcal{D}_X$-module M is $\mathcal{O}_X$-coherent if and only if it is an integrable connection.

If $E \neq 0$, then $car(E, \nabla) = X \subset T^*X$ (the zero section). $car(0) = \emptyset$.

Moduli for connections (E, ∇)
Simpson ($\sim$1989) (published in 1994).

Simpson proved boundedness (for fixed rank), and constructed a quasi-projective moduli scheme. Points of the moduli are Jordan-Holder classes (semi-simplifications) of (E, ∇).

Sketch of boundedness:

Complex Chern classes $c_i(E) = 0$ for $i \geq 1$.
Hilbert polynomial $\chi(E(m)) = rank(E)\chi(\mathcal{O}_X(m))$.

If $\mathcal{F}$ is a coherent $\mathcal{O}$-submodule of E with $\mu(\mathcal{F})$ maximal, then $F^r\mathcal{D}_X\mathcal{F} \subset E$ (where $r = rank(E)$) is $\mathcal{O}$-coherent and generically a $\mathcal{D}_X$-submodule, with $\overline{\mu}F^r\mathcal{D}_X$ not too much less. Its $\mathcal{O}_X$-saturation $(F^r\mathcal{D}_X)'$ has $\overline{\mu}$ even higher. But this is an integrable connection, so has $c_1 = 0$.

Hence $\mu(\mathcal{F})$, and so $\overline{\mu}(E)$, is bounded above. Now apply **Maruyama's Boundedness Theorem** to conclude.

Regular meromorphic connections

$Y \subset X$ normal crossing divisor. $j : X - Y \hookrightarrow X$. (E', ∇') flat connection on $X - Y$ in algebraic category. $M = j_* E'$ naturally $\mathcal{D}_X$-module, called **meromorphic connection**.

If $M \neq 0$, then $car(M) = C_{X,Y}$ which has dimension d. Hence meromorphic connections are **holonomic**.

Regularity condition: (E', ∇') must have an **algebraic** prolongation (E, ∇), with E a vector bundle on X together with a **logarithmic connection** $\nabla : E \to \Omega^1_X \langle \log Y \rangle \otimes E$.

$\Omega^1_X \langle \log Y \rangle$ is the locally free $\mathcal{O}_X$-module of logarithmic 1-forms, equal to $(T_X \langle \log Y \rangle)^\vee$, where $T_X \langle \log Y \rangle \subset T_X$ consists of tangent vector fields that preserve the ideal sheaf $I_Y \subset \mathcal{O}_X$.

Caution: A holomorphic prolongation (which is not necessarily algebraic as a **prolongation**) always exist, by **Deligne construction**.

Deligne construction Let $Y \subset X$ be normal crossing divisor. To any monodromy representation $\pi_1(X - Y) \to GL_n$, there is functorially associated a logarithmic connection (E, ∇) with all residual eigenvalues in any chosen fundamental domain $\Sigma \subset \mathbb{C}$ for the exponential map $z \mapsto \exp(2\pi i z)$ from $\mathbb{C}$ to $\mathbb{C} - \{0\}$.

In local coordinate description, the residue matrix R is the logarithm of the local monodromy, with eigenvalues of R in Σ.

If Σ is chosen to be the strip $0 \leq \Re(z) < 1$, then we exactly get all the Deligne connections.

Moduli for semistable logarithmic connections ([N. 1993]).

Regular meromorphic connections on (X, Y) do not even form an (Artin) algebraic stack: the diagonal is not quasi-compact.

Logarithmic connections form an algebraic stack.

Definition of **semi-stable** logarithmic connection (E, ∇): Every $\mathcal{O}_X$-coherent ∇-invariant subsheaf $F \subset E$ satisfies the semi-stability inequality between reduced Hilbert polynomials.

$$P(F, m)/rank(F) \leq P(E, m)/rank(E)$$

Fix Hilbert polynomial. Semistable are bounded. Quasi-projective moduli via Simpson's method.

The passage from logarithmic connections to regular meromorphic connections is infinitesimally rigid if residual eigenvalues are 'good'.

Defects of [N. 1993] definition of semistability of logarithmic connections:
(1) The link between residues and degree is ignored. So semistability 'shrinks' too much.
(2) The link with Narasimhan-Seshadri Theorem is lost.

Common Observation: There is a **natural parabolic structure** on any Deligne construction. The parabolic degree is zero. Hence automatically par-μ-semistable.

For moduli construction, we take parabolic Gieseker semi-stability. Combine methods of **Simpson** and **Maruyama-Yokogawa**.

This approach is better also for treating more general regular holonomic $\mathcal{D}_X$-modules (as we will see later in this talk).

Price to pay

(1) We have to fix eigenvalues of residue.

(However, note that this is **automatic** along components Y_a for which the **universal topological degree** $\delta(N_a)$ of normal bundle is non-zero, as the local loop τ_a around Y_a maps to $\delta(N_a)$ th roots of 1 under any $\pi_1(X-Y) \to \mathbb{C}^\times$).

$c_1(N_a) \cap H_2(Y_a; \mathbb{Z}) = \delta(N_a) \cdot \mathbb{Z}$

(2) Moduli is for parabolic Gieseker semistable, which need not coincide with par-μ-semistable.

(However, when all components of the divisor have self-intersection zero, or when the residue is nilpotent, par-Gieseker and par-μ coincide.)

Some Parabolic History

Parabolic bundles on curves, parabolic semistability: **Seshadri** ($\sim$1970).

Moduli construction: **Mehta and Seshadri** (1980). Generalised Narasimhan-Seshadri theorem to unitary monodromy on punctured curves.

Parabolic bundles, semistability and moduli in higher dimensions: **Maruyama and Yokogawa** (1992). Parabolic Higgs bundles: **Yokogawa** (1993).

The **Narasimhan-Seshadri-Donaldson-Hitchin-Corlette-Simpson-Biquard** correspondence between local systems and 'bundles'.

Simpson and Biquard correspondence is from par-μ-polystable logarithmic connections to par-μ-polystable logarithmic Higgs bundles.

Poincaré residue map

Let $(x_1, \ldots, x_n)$ be local coordinates on X, with Y locally defined by $x_1 x_2 \cdots x_m = 0$.
Then $\Omega^1_X\langle \log Y\rangle$ is locally free with basis $dx_1/x_1, \ldots, dx_m/x_m$, $dx_{m+1}, \ldots, dx_n$.

$\widetilde{Y} \to Y$: the normalisation of Y.

The Poincaré residue map
res : $\Omega^1_X\langle \log Y\rangle|_Y \to \mathcal{O}_{\widetilde{Y}}$ is defined by $dx_a/x_a \mapsto 1$ for $a \leq m$ and $dx_b \mapsto 0$.

Short exact sequence
$0 \to \Omega^1_X \to \Omega^1_X\langle \log Y\rangle \to \mathcal{O}_{\widetilde{Y}} \to 0$

Under the connecting map
$H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) \to H^1(X, \Omega^1_X)$: $1 \mapsto -[Y]$

This links residue of ∇ to Chern classes $c_i(E)$.

Residue for (E, ∇)

∇ is not $\mathcal{O}_X$-linear, but the composite

$E \xrightarrow{\nabla} \Omega^1_X\langle \log Y\rangle \otimes E|_Y \xrightarrow{\text{res}} E|_{\widetilde{Y}}$ is $\mathcal{O}_X$-**linear**.

Pullback to $\widetilde{Y}$ defines res(∇) $\in End(E|_{\widetilde{Y}})$

Local coordinate description:

Let e_i be a local basis for E. Then $\nabla(e_i) =$

$\sum_j \left(\sum_{a\leq m} R^j_{i,a}\frac{dx_a}{x_a} + \sum_{b>m} \Gamma^j_{i,b} dx_b\right) \otimes e_j$

where $R^j_{i,a}$ and $\Gamma^j_{i,b}$ are local sections of $\mathcal{O}_X$. The matrices $(R^j_{i,a}|_{Y_a})$ define res(∇) on $E|_{Y_a}$, where Y_a is locally defined by $x_a = 0$.

Connection with Newton classes $N_p(E)$

Well-known fact (see [Esnault-Viehweg 1986]):
Residue $\mapsto$ Atiyah obstruction $\mapsto$ Newton classes.

For $p \geq 0$, by definition $N_p(E) = \sum_{1 \leq i \leq r} (\gamma_i)^p$ where $r = rank(E)$ and γ_i are the Chern roots of E.

$Y = Y_1 \cup \ldots \cup Y_m$ with each Y_a smooth, crossing normally. $R_a = \text{res}(\nabla)|_{Y_a}$. Then $N_p(E)$ equals

$$(-1)^p \sum_{q_1+\ldots+q_m=p} Tr(R_1^{q_1} \cdots R_m^{q_m})[Y_1]^{q_1} \cdots [Y_m]^{q_m}$$

where $[Y_a] = c_1(\mathcal{O}_X(Y_a)) \in H^2(X^{an}, \mathbb{C})$.

Hence $ch(E) = r + N_1 + N_2/2 + N_3/3! + \ldots$ is determined by the residue.

Natural split parabolic structure

$Y \subset X$ normal crossing divisor. (E, ∇) a vector bundle with a logarithmic connection. The characteristic polynomial of the residue $R_a \in End(E|_{Y_a})$ has constant coefficients as Y_a is connected.

Definition (E, ∇) is a **Deligne connection** if real parts of eigenvalues of $R_a \in End(E|_{Y_a})$ all lie in the interval $[0, 1)$.

Let $0 \leq \alpha_{a,1} \leq \ldots \alpha_{a,p} < 1$ be real parts along Y_a. Then $E|_{Y_a}$ gets direct sum decomposition

$$E|_{Y_a} = E_{a,1} \oplus \ldots \oplus E_{a,p}$$

where $E_{a,i}$ is the direct sum of generalised eigensubbundles of R_a for all eigenvalues λ with $\Re(\lambda) = \alpha_{a,i}$.

Filtration $F_{a,i} = \oplus_{j \geq i} E_{a,i}$ of $E|_{Y_a}$ with weights $\alpha_{a,i}$ is the natural parabolic structure on E.

Parabolic Hilbert polynomial

Fix very ample $\mathcal{O}_X(H)$. Let E be a Deligne connection. Following Maruyama-Yokogawa,

$$par\,\chi(E,m) = \chi(E(-Y),m) + \sum \alpha_{a,i}\chi(E_{a,i},m)$$

When there is only one parabolic weight equal to 0, then note that $par\,\chi(E,m) = \chi(E(-Y),m)$.

By Riemann-Roch $\chi(E(-Y),m) =$

$\frac{r(E)[H]^d}{d!}m^d + \frac{(r(E)c_1(X)/2 + c_1(E(-Y)))[H]^{d-1}}{(d-1)!}m^{d-1} +$ ... (lower order terms in m).

$\sum \alpha_{a,i}\chi(E_{a,i},m) = \sum \alpha_{a,i}\frac{r(E_{a,i})[Y_a][H]^{d-1}}{(d-1)!}m^{d-1} +$ ...

$\sum \alpha_{a,i} r(E_{a,i})[Y_a] = \sum tr(R_a)[Y_a] = -c_1(E)$

Therefore $par\,\chi(E,m) =$

$\frac{r(E)[H]^d}{d!} m^d + \frac{r(E)(c_1(X)/2 - [Y])[H]^{d-1}}{(d-1)!} m^{d-1} + \ldots$ (lower order terms in m).

Hence parabolic degree is always zero and so we have **automatic** parabolic μ-semistability.

Caution The coefficients in degrees $\leq d-2$ depend on data involving residual eigenvalues and their intersection multiplicities. So parabolic Gieseker semistability is not automatic except in special cases – say when each $[Y_a][Y_b] = 0$ or when along each Y_a there is exactly one parabolic weight $\Re(\lambda_{a,i})$ (e.g. all R_a nilpotent).

Definition: Parabolic semistability for a Deligne connection E : for each nonzero ∇-invariant vector subbundle F

$$par\,\chi(F,m)/r(F) \leq par\,\chi(E,m)/r(E)$$

Parabolic stability: strict inequality.

Because of **strong local freeness**, no condition on other $\mathcal{O}$-coherent ∇-subsheaves.

Sub-connections of E have only **finitely many** possible Hilbert polynomials and parabolic Hilbert polynomials.

Openness of semistability and stability by the usual Quot scheme argument of **Narasimhan and Ramanathan**.

Moduli construction for Deligne connections

Given (X, Y) with $Y = \cup_a Y_a$, and rank n, we fix

(1) along each Y_a, eigenvalues (with multiplicities) for residues R_a with real parts in $[0, 1)$,

(2) along each connected component of intersection $Y_{a_1} \cap \ldots \cap Y_{a_r}$, fix ranks of intersections of generalised eigen-subbundles of residues R_{a_i}.

This fixes Newton classes of any E with the above data. Boundedness.

Combination of Simpson's method for Λ-modules with Bhosle-Maruyama-Yokogawa for parabolic bundles converts the moduli problem into a quotient problem and into a GIT problem.

By boundedness there exists N such that $E(m)$ has all higher cohomologies zero for $m \geq N$, and is generated by global sections.

Locally universal family

Fix rank r and residual eigenvalues with multiplicities of the Deligne connection.

$\mathcal{D}_X\langle\log Y\rangle \subset \mathcal{D}_X$ consists of operators which preserve $I_Y \subset \mathcal{O}_X$. Logarithmic connections same as $\mathcal{O}_X$-coherent $\mathcal{D}_X\langle\log Y\rangle$-modules locally free over $\mathcal{O}_X$.

Boundedness $\Rightarrow$ $\exists$ a locally universal family parametrised by a locally closed subscheme C of the Quot scheme of $\mathcal{O}_X$-coherent quotients

$$F^{r+1}\mathcal{D}_X\langle\log Y\rangle(-N)^{\oplus P_E(N)} \to \mathcal{F}$$

N chosen such that for $m \geq N$, $E(m)$, $E_{a,\alpha}(m)$ generated by global sections and higher H^i vanish, also for I_Y times or I_x times the sheaves for $x \in X$ and same for $\mathcal{D}_X\langle\log Y\rangle$-subquotient bundles of E and $E_{a,\alpha}$.

$C \subset Quot(F^{r+1}\mathcal{D}_X\langle\log Y\rangle(-N)^{\oplus n} \to \mathcal{F})$ is locally closed, defined by the following conditions (where $n = P_E(N)$):

(i) $\mathcal{F}$ is locally free (open condition).

(ii) $\mathcal{O}_X(-N)^{\oplus n} \to \mathcal{F}$ is surjective, induces isomorphism $\mathbb{C}^n \to H^0(\mathcal{F}(N))$ (open condition).

(iii) $F^1 \otimes \mathcal{O}_X(-N)^{\oplus P_E(N)} \to \mathcal{F}$ factors via $F^1 \otimes \mathcal{F}$ giving $F^1 \otimes \mathcal{F} \to \mathcal{F}$, and product $F^i \otimes F^j \to F^{i+j}$ for $i + j \leq r + 1$ is respected (closed condition, makes $\mathcal{F}$ a $\mathcal{D}_X\langle\log Y\rangle$-module.)

(iv) Residual eigenvalues correct (closed condition).

The scheme C in invariant under the action of SL_n and a good quotient $C//SL_n$, **if it exists**, is the coarse moduli. $C^{par\, ss}//SL_n$ exists.

Parabolic polarisation and GIT quotient

Embed C into a product of Quot schemes $Q_0 \times \prod_{a,i} Q_{a,i}$ for quotients of types $F^{r+1}(-N)^{\oplus n} \to E$ and $F^{r+1}(-N)^{\oplus n} \to E|_Y/F_{a,i+1}$ where $F_{a,i} = \oplus_{j \geq i} E_{a,i}$ is the parabolic filtration.

This embeds into a product of a Gieseker space and Grassmanians $Z \times \prod_{a,i} Gr_{a,i}$ following Bhosle-Maruyama-Yokogawa.

SL_n-equivariant.

Linearization : Assume each $\alpha_{a,i}$ is rational.

Give $Z \times \prod_{a,i} Gr_{a,i}$ polarisation ($par\chi(E,N)$, $\epsilon_{a,i}$), where $\epsilon_{a,i} = \alpha_{a,i+1} - \alpha_{a,i}$ or $1 - \alpha_{a,\ell(a)}$ for the largest $\ell(a)$.

Then parabolic (semi-)stability exactly corresponds to GIT (semi-)stability for all large N.

Strong local freeness

Lemma E Deligne connection, $\mathcal{F} \subset E$ sub $\mathcal{D}_X\langle \log Y \rangle$-module. If $E/\mathcal{F}$ is torsion-free then $E/\mathcal{F}$ is locally free, that is, $\mathcal{F}$ is a vector subbundle of E.

Proof $\mathcal{M} =$ local system on $X - Y$ defined by $E|_{X-Y}$. $\mathcal{F}|_{X-Y}$ is locally free, so defines a sub local system $\mathcal{L} \subset \mathcal{M}$. By Deligne construction gives a subbundle $\overline{F} \subset E$ with $\mathcal{F}|_{X-Y} = \overline{F}|_{X-Y}$. So $\mathcal{F} \subset \overline{F}$ and $\overline{F}/\mathcal{F}$ is torsion, while $\overline{F}/\mathcal{F} \subset E/\mathcal{F}$ which is torsion free. Hence $\mathcal{F} = \overline{F}$.

Application to prove that GIT-s.s. orbit limits under 1-parameter subgroups of SL_n correspond to par-S-filtrations by subconnections. Points of moduli are parabolic polystable Deligne connections with the given characteristic polynomials $f_a(\lambda)$ of residues R_a.

Pre-$\mathcal{D}$-modules ([N. and Claude Sabbah 1996, N. 1999])

Regular holonomic $\mathcal{D}_X$ modules are not in general $\mathcal{O}_X$-coherent.

$\mathcal{O}_X$-coherent description of meromorphic connections is by logarithmic connections.

$\mathcal{O}_X$-coherent description of more general regular holonomic $\mathcal{D}_X$-modules is by pre-$\mathcal{D}$-modules.

The definition of semi-stability in our earlier papers is far too restrictive.

We now define a natural parabolic semi-stability and construct moduli $\mathcal{M}$. Points of $\mathcal{M}$ are S-equivalence classes.

There is a moduli $\mathcal{N}$ of corresponding perverse sheaves by [N. 1999]. The Riemann-Hilbert correspondence defines a complex analytic morphism $\mathcal{M} \to \mathcal{N}$. This is a local homeomorphism at stable points of $\mathcal{M}$.

When Y is smooth : some preliminaries

Lemma Let F be an $\mathcal{O}_X$-coherent $\mathcal{D}_X\langle\log Y\rangle$-module which is schematically supported on Y. Then each $F_a = F|_{Y_a}$ is locally free.

Proof Choose local trivialization for N_Y. Then locally we get flat connections on F.

Sheaf of rings $\mathcal{D}_X\langle\log Y\rangle|_Y$
As $\mathcal{D}_X\langle\log Y\rangle$ preserves I_Y, we have $I_Y\mathcal{D}_X\langle\log Y\rangle = \mathcal{D}_X\langle\log Y\rangle I_Y$. Hence we get a 2-sided ideal. Quotient ring is $\mathcal{D}_X\langle\log Y\rangle|_Y$.

Euler operator $x\,\partial/\partial x|_{x=0}$
$0 \to \mathcal{O}_Y \to T_X\langle\log Y\rangle|_Y \to T_Y \to 0$
Image of 1 defines a section e of $T_X\langle\log Y\rangle|_Y$ and of $\mathcal{D}_X\langle\log Y\rangle|_Y$. **Basic fact** The Euler operator e is **central** in $\mathcal{D}_X\langle\log Y\rangle|_Y$.

Residue For any $\mathcal{D}_X\langle\log Y\rangle|_Y$-module F, the endomorphism $\mathbf{e} : F \to F$ will be denoted by res(F).

As $\mathbf{e}$ is central, res(F) is $\mathcal{D}_X\langle\log Y\rangle|_Y$-linear. Hence eigenvalues are constants (even without compactness of Y).

Lemma Let $\alpha_{a,i}$ be eigenvalues of res(F), and $F = \oplus F_{a,i}$ generalised eigensubbundles, then the Chern character of $F_{a,i}$ is given by

$$ch(F_{a,i}) = r_{a,i}\,\mathrm{exp}(-\alpha_{a,i}c_1(N_Y))$$

Proof The pull-back of F to the total space of N_Y becomes a logarithmic connection on (N_Y, Y).

Lemma $F_{a,i}$ is a semi-stable $\mathcal{D}_X\langle\log Y\rangle$-module (both Gieseker and μ).

Parabolic $\mathcal{D}$-modules on (X, Y)

Simplest case: Let Y be smooth, but not necessarily connected.

A **par-$\mathcal{D}$-module** on (X, Y) is a tuple (E, F, s, t) where

(i) E is a Deligne connection on (X, Y).

(ii) F is an $\mathcal{O}_X$-coherent $\mathcal{D}_X\langle\log Y\rangle$-module on Y such that any eigenvalue of res(F) has real part in $[0, 1)$.

(iii) $s : F \to E|_Y$ and $t : E|_Y \to F$ are $\mathcal{D}_X\langle\log Y\rangle$-linear with $st = \text{res}(E)$ and $ts = \text{res}(F)$.

Fact s and t are isomorphisms on generalised eigensubbundles of res for eigenvalues $\lambda \neq 0$.

Relation with $\mathcal{D}$-modules
(case when Y is smooth.)

(E, F, s, t) is a par-$\mathcal{D}$-module on (X, Y).

$\mathcal{D}_X\langle \log Y\rangle$-submodule $E \oplus_s F \subset E \oplus F$ consists of all (u, v) with $u|_Y = s(v)$. Makes sense as s is $\mathcal{D}_X\langle \log Y\rangle|_Y$-linear.

Let $M_0 = E$, $M_1 = \mathcal{O}_X(Y) \otimes_{\mathcal{O}_X} (E \oplus_s F)$.
$\mathcal{D}_X\langle \log Y\rangle$-linear inclusion $M_0 \hookrightarrow M_1$ defined by

$$u \mapsto x^{-1} \otimes (xu, 0)$$

Define connection operator $\nabla : M_0 \to \Omega^1_X \otimes M_1$ by

$$\nabla_\eta(u) = x^{-1} \otimes (\, (x\eta)(u),\, \eta(x)t(u|Y)\,)$$

for $\eta \in T_X$. This is compatible with given $\mathcal{D}_X\langle \log Y\rangle$-structures.

Finally, define M to be the left $\mathcal{D}_X$ module which is the quotient of $\mathcal{D}_X \otimes_{\mathcal{D}_X\langle \log Y\rangle} M_1$ by its submodule generated by all elements of the type $\eta \otimes u - 1 \otimes \eta(u)$.

Proposition The $\mathcal{D}_X$-module M is regular holonomic, with characteristic variety $C_{X,Y}$.

Backward passage via **V-filtration** $\ldots V^i(M) \subset V^{i+1}(M)\ldots$, which is based on the Deligne construction. $E = V^0(M)$, $F = V^1(M)/V^0(M)$, s and t locally induced by x and $\partial/\partial x$.

Infinitesimal rigidity (a consequence of [N. 1993] for Deligne connections):

A par-$\mathcal{D}$-module (E, F, s, t) does not admit any nontrivial infinitesimal deformation such that the associated $\mathcal{D}_X$-module is constant.

Semistability and moduli
(Case when Y is smooth)

We say (E, F, s, t) is par-semistable if E is a par-semistable Deligne connection.

The generalised eigensubbundles $F_0 \subset F|_Y$ for eigenvalue 0 are semistable $\mathcal{D}_X\langle \log Y\rangle|_Y$-modules, as already seen.

There exist local universal schemes R and R' for these, with actions of G and G', with good quotients (the moduli spaces). The extra data (s, t) is parametrised by a scheme U, **affine over** $R \times R'$. Natural lift of $G \times G'$-action to U. By **Ramanathan's lemma** a good quotient $\mathcal{M}$ exists.

$\mathcal{M}$ is coarse moduli for (E, F, s, t).

Points of the moduli
(when Y is smooth).

The closed points of the moduli correspond to isomorphism classes of $\mathcal{D}_X$-modules of the form

$$j_{!+}(E|_{X-Y}) \oplus i_+(F)$$

where $j : X - Y \hookrightarrow X$ is the inclusion, E is a par-polystable Deligne connection on (X, Y), $j_{!+}(E|_{X-Y})$ is the **minimal prolongation** of the $\mathcal{D}_{X-Y}$-module $E|_{X-Y}$ to a $\mathcal{D}_X$-module, $i : Y \to X$ is the closed embedding, F is a semisimple non-singular connection on Y, and $i_+(F)$ denotes its direct image in the category of $\mathcal{D}$-modules.

Let E' be kernel of $E \to coker(\text{res}(E))$ (this is an **elementary transform** in the sense of Maruyama). Then $j_{!+}(E|_{X-Y})$ is the $\mathcal{D}_X$-submodule of $j_*(E|_{X-Y})$ generated by $\mathcal{O}_X(Y) \otimes E'$.

General case of normal crossing divisor

A par-$\mathcal{D}$-module on (X, Y) consists of data (E_i, s_i, t_i), where

E_d Deligne connection on (X, Y), $d = dim(X)$.

$\widetilde{Y} = \coprod Y_a$ the normalisation of Y.

E_{d-1} is a $\mathcal{D}_X\langle \log Y\rangle|_{\widetilde{Y}}$-module on $\widetilde{Y}$, locally free as $\mathcal{O}$-module, such that when $Y_a \cap Y_b \neq \emptyset$, the residual eigenvalues of $E_{d-1,a}$ along Y_b have real parts in $[0, 1)$.

E_{d-2} is similar stuff on the normalisation of $sing(Y)$, etc.

The (s, t) are $\mathcal{D}(log)$-linear and satisfy conditions $st =$ res and $ts =$ res, with commutativity conditions given by

$s_{i,a}s_{i-1,b} = s_{i,b}s_{i-1,a}$,

$t_{i-1,a}t_{i,b} = t_{i-1,b}t_{i,a}$

$s_{i-1,a}t_{i,b} = t_{i+1,b}s_{i,a}$ for $a \neq b$.

In a polydisk, this is the **hypercube diagram** of **Galligo, Granger, Maisenobe** (1985). In terms of local coordinates on the polydisk, and with E_i defined using the V-filtration, we have $s_a = x_a$ and $t_a = \partial/\partial x_a$.

Passage to $\mathcal{D}_X$-modules. Infinitesimal rigidity.

Semistability and moduli
(General case)

We say (E_i, s_i, t_i) is par-semistable if E_d is a par-semistable Deligne connection, and for $i \leq d-1$ the generalised eigensubbundles $E_{i,0} \subset E_i$ of the residue are par-semistable logarithmic modules.

(Other eigenvalues do not matter, as s, t are isomorphisms for generalised eigensubbundles for $\lambda \neq 0$.)

There exist local universal schemes R_i for these, with actions of G_i with good quotients the moduli spaces for par-semistable $\mathcal{D}(log)$-modules. The extra data (s, t) is parametrised by a scheme U, affine over $\prod_i R_i$, with lift of $\prod_i G_i$-action.

Again by Ramanathan's lemma a good quotient $\mathcal{M}$ exists, which is the coarse moduli for (E_i, s_i, t_i).

S-filtration: maximal filtration by subobject (E_i', s_i', t_i') with same $par\chi(m)/rank$. Strong local freeness holds for the E_i. Points of moduli $\mathcal{M}$ are S-equivalence classes.

A Riemann-Hilbert morphism $\mathcal{M} \to \mathcal{N}$ to moduli $\mathcal{N}$ of perverse sheaves is defined by analytic properties of a good quotient $U \to \mathcal{M}$.